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PROPAGATION OF LONGITUDINAL AND TRANSVERSE WAVES IN A MULTIMODULUS ELASTIC MEDIUM

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A problem of propagation of longitudinal and transverse waves in a multimodulus elastic isotropic medium is considered. In the model used, the medium is described by a potential depending on three invariants of strains, which allows the influence of preliminary deformation of the medium on the longitudinal and transverse velocities to be taken into account.

Key words: *internal energy, potential, multimodulus model, longitudinal and transverse velocities.*

1. In considering adiabatic processes of elastic deformation, it is usually postulated that internal energy depends on invariants of strain measures. In the general form, the tensor relation between the stresses and strains for a nonlinearly elastic isotropic medium can be written as

$$\sigma_{ij} = K_0\delta_{ij} + K_1\varepsilon_{ij} + K_2\varepsilon_{in}\varepsilon_{nj}.$$

Here K_0, K_1, and K_2 are the functions of three invariants of the strain tensor $I_1 = \varepsilon_{mm}$, $I_2 = \varepsilon_{mn}\varepsilon_{nm}$, and $I_3 = \varepsilon_{mn}\varepsilon_{np}\varepsilon_{pm}$ (summation is performed over repeated indices from 1 to 3).

If a potential $W = W(I_1, I_2, I_3)$ exists, we obtain

$$K_0 = \frac{\partial W}{\partial I_1}, \qquad K_1 = 2\frac{\partial W}{\partial I_2}, \qquad K_2 = 3\frac{\partial W}{\partial I_3}.$$

One of the methods most frequently used in the classical elasticity theory is the choice of the potential in the form of the expansion of internal energy into a Taylor series with respect to the initial state with retaining a certain number of terms. If the second, third, and fourth powers of strains are retained, we have nine terms. The constants at the corresponding powers of strains are called the Lame constants of the second, third, and fourth order. Other notations for the potential can also be used, for instance, those in Landau and Murnaghan nonlinear models. The relations between the constants in these models and the values of some constants can be found in [1]. These models can also be used to describe the behavior of materials with different resistances to tension and compression. It is assumed that these moduli can become different, in particular, owing to internal damages of the material and appearance of cracks, which makes the elastic characteristics and the character of propagation of elastic waves in the material depend on the type of the stress state. The qualitative characteristic responsible for the type of the strain state is the third invariant of the strain tensor, while the first and second invariants characterize the changes in the volume and shape. The difference in resistances to tension and compression can be taken into account to a certain extent by introducing a term depending on $\xi = I_1\sqrt{I_2}$ into the potential. Potentials of this type were considered in [2, 3]. The potential in [3] was chosen in the form

$$W(\varepsilon_{i,j}) = 0.5[K + \varphi(\xi)]I_1^2 + [G + \psi(\xi)](I_2 - I_1^2/3),$$

where the functions φ and ψ are related as $3\xi^2\varphi' + 2(3-\xi^2)\psi' = 0$. In this case, however, experiments on tension and compression with measurements of longitudinal and transverse strains yield four relations for determining a smaller

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number of constants. As a result, an additional condition appears, which restricts the area of model applicability. A similar model was considered in [4] with the use of an elastic potential in stresses in the form

$$W = 0.5[1 + f(\xi)](A + B\xi^2)\sigma_i^2,$$

where $\sigma_i^2 = (3/2)S_{ij}S_{ij}$ is the second invariant of the stress deviator, $\xi = \sigma/\sigma_i$, $\sigma = (1/3)\sigma_{ii}$, and A and B are constants.

Matchenko and Tolokonnikov [5] made an attempt to take into account the influence of the third invariant of the stress tensor on the type of the strain state; this invariant was introduced in the form $\cos 3\varphi$ (φ is the angle of the stress state type, which depends on the third invariant of the deviator and on the degree of octahedral deformation). The form of the potential used in [6] was more complicated.

The potential proposed in [7, 8] seems to have the most convenient form for calculations:

$$W = \frac{1}{2}\,\lambda I_1^2 + \mu I_2 - \gamma I_1\sqrt{I_2} + \alpha\,\frac{I_1^3}{\sqrt{I_2}} + \delta\,\frac{I_3}{\sqrt{I_2}}.$$

Derivation of this potential and constitutive equations on the basis of this potential was described in [2]. This potential has a tensor-nonlinear term, which take into account the interaction between the second and third invariants of strains. The term $\alpha I_1^3/\sqrt{I_2}$ can usually be omitted, because its influence is taken into account to a certain extent by the term $\gamma I_1\sqrt{I_2}$. In what follows, therefore, we consider the potential with $\alpha = 0$.

The potential with $\alpha = \delta = 0$ was considered in [9]. It was demonstrated there that, for the potential chosen in this form, the velocity of wave propagation depends on the propagation direction, stress state, and fissuring characterized by the term $\gamma I_1\sqrt{I_2}$.

2. Let us consider a potential

$$W = \frac{1}{2}\,\lambda I_1^2 + \mu I_2 - \gamma I_1\sqrt{I_2} + \delta\,\frac{I_3}{\sqrt{I_2}} \tag{2.1}$$

with $W = 0$ and $\partial W/\partial\varepsilon_{ij} = 0$ for $\varepsilon_{ij} = 0$. Then, we obtain

$$\sigma_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} = \delta_{ij}\,\frac{\partial W}{\partial I_1} + 2\varepsilon_{ij}\,\frac{\partial W}{\partial I_2} + 3\varepsilon_{ki}\varepsilon_{jk}\,\frac{\partial W}{\partial I_3} = \Big(\lambda - \frac{\gamma}{\xi}\Big)I_1\delta_{ij} + (2\mu - \gamma\xi - \delta\eta\xi)\varepsilon_{ji} + 3\varepsilon_{ki}\varepsilon_{jk}\,\frac{\delta}{\sqrt{I_2}}, \tag{2.2}$$

where $\xi = I_1/\sqrt{I_2}$ and $\eta = I_3/(I_1 I_2)$ are the relative invariants of the strain state.

Denoting $\lambda^* = \lambda - \gamma/\xi$ and $2\mu^* = 2\mu - \gamma\xi - \delta\eta\xi$, we obtain the notation for stresses with nonlinear Lamé parameters and a nonlinear term, which is commonly accepted in the elasticity theory. Using λ^* and μ^*, we can apply conventional formulas and determine the moduli G^*, K^*, ν^*, and E^* similar to those used in the linear elasticity theory.

Inversion of Eq. (2.2) is complicated by the presence of a nonlinear term. For a potential of a more general form (with allowance for the term with α), inversion was performed in [2] with allowance for the results obtained in [10].

3. The constants λ, μ, γ, and δ are determined from experiments on uniaxial tension and compression with measurements of transverse strains. In the case of tension, we have $\sigma_{11}^{\text{tens}} = \sigma$, $\sigma_{ij}^{\text{tens}} = 0$ $(ij \neq 11)$, $\varepsilon_{11}^{\text{tens}} = \varepsilon$, and $\varepsilon_{22}^{\text{tens}} = \varepsilon_{33}^{\text{tens}} = -\nu_{\text{tens}}\varepsilon$. Similarly, in the case of compression, we obtain $\sigma_{11}^{\text{compr}} = -\sigma$, $\sigma_{ij}^{\text{compr}} = 0$ $(ij \neq 11)$, $\varepsilon_{11}^{\text{compr}} = -\varepsilon$, and $\varepsilon_{22}^{\text{compr}} = \varepsilon_{33}^{\text{compr}} = \nu_{\text{compr}}\varepsilon$. The elastic characteristics determined in experiments on tension and compression are indicated by "tens" and "compr," respectively. Thus, we obtain

$$\begin{gathered} E_{\text{tens}} = \sigma_{11}^{\text{tens}}/\varepsilon_{11}^{\text{tens}} = (\lambda - \gamma/\xi_{\text{tens}})(1 - 2\nu_{\text{tens}}) + 2\mu - \gamma\xi_{\text{tens}} - \delta\eta_{\text{tens}}\xi_{\text{tens}} + 3\delta\Big/\sqrt{1 + 2\nu_{\text{tens}}^2}, \\ (\lambda - \gamma/\xi_{\text{tens}})(1 - 2\nu_{\text{tens}}) - 2\mu - \gamma\xi_{\text{tens}} - \delta\eta_{\text{tens}}\xi_{\text{tens}}\nu_{\text{tens}} + 3\delta\nu_{\text{tens}}^2\Big/\sqrt{1 + 2\nu_{\text{tens}}^2} = 0. \end{gathered} \tag{3.1}$$

Similarly, in the case of compression, we have

$$\begin{gathered} E_{\text{compr}} = \sigma_{11}^{\text{compr}}/\varepsilon_{11}^{\text{compr}} = (\lambda - \gamma/\xi_{\text{compr}})(1 - 2\nu_{\text{compr}}) + 2\mu - \gamma\xi_{\text{compr}} - \delta\eta_{\text{compr}}\xi_{\text{compr}} - 3\delta\Big/\sqrt{1 + 2\nu_{\text{compr}}^2}, \\ -(\lambda - \gamma/\xi_{\text{compr}})(1 - 2\nu_{\text{compr}}) + (2\mu - \gamma\xi_{\text{compr}} - \delta\eta_{\text{compr}}\xi_{\text{compr}})\nu_{\text{compr}} + 3\delta\nu_{\text{compr}}^2\Big/\sqrt{1 + 2\nu_{\text{compr}}^2} = 0. \end{gathered} \tag{3.2}$$

Determining the constants λ, μ, γ, and δ reduces to solving system (3.1), (3.2).

TABLE 1

Calculated Constants λ, μ, γ, and δ for Three Materials

Material	E_{tens}, kg/mm^2	ν_{tens}	E_{compr}, kg/mm^2	ν_{compr}	λ, GPa	μ, GPA	γ, GPa	δ, GPa
St. 40 steel	205.79	0.29	211.79	0.28	107.74	81.24	−3.61	−3.59
SCh 12-28 cast iron	91.41	0.22	121.87	0.27	40.61	42.98	1.52	−7.40
AL-2 silumin	66.93	0.34	73.42	0.39	69.27	25.80	45.67	−0.36

Ambartsumyan [11] gave the values of Young's moduli and Poisson's ratios obtained in experiments on uniaxial tension and compression for three materials: St. 40 steel (normalized), SCh 12-28 cast iron, and AL-2 silumin. The constants λ, μ, γ, and δ for these three materials are calculated in the present work by Eqs. (3.1) and (3.2); there values are listed in Table 1. Rather large differences in the values of ν_{tens} and ν_{compr} for cast iron and silumin and in the values of E_{tens} and E_{compr} for cast iron should be noted.

4. Let us assume that the material is in a uniform stress state. We denote the equilibrium values of strains and stresses by ε_{ij}^0 and σ_{ij}^0. The values I_i^0 $(i = 1, 2, 3)$, ξ_0, η_0, $\lambda_0^* = \lambda - \gamma/\xi_0$, and $2\mu_0^* = 2\mu - \gamma\xi_0 - \delta\eta_0\xi_0$ correspond to these strains and stresses.

Let $\hat{\varepsilon}_{ij}$ and $\hat{\sigma}_{ij}$ be additives induced by dynamic actions, which are small as compared with the equilibrium strains and stresses. Then, we obtain $\varepsilon_{ij} = \varepsilon_{ij}^0 + \hat{\varepsilon}_{ij}$ and $\sigma_{ij} = \sigma_{ij}^0 + \hat{\sigma}_{ij}$.

As the additives are small, we have

$$\hat{I}_1 = \hat{\varepsilon}_{ii}, \qquad \hat{I}_2 = 2\hat{\varepsilon}_{ij}\varepsilon_{ji}^0, \qquad \hat{I}_3 = 3\hat{\varepsilon}_{ij}\varepsilon_{jk}^0\varepsilon_{ki}^0,$$

$$\hat{\xi} = \xi_0(\hat{\varepsilon}_{ii}/I_1^0 - \hat{\varepsilon}_{ij}\varepsilon_{ij}^0/I_2^0), \qquad \hat{\eta} = \eta_0(\hat{I}_3/I_3^0 - \hat{I}_2/I_2^0 - \hat{I}_1/I_1^0).$$

For the stresses $\hat{\sigma}_{ps}$, we obtain

$$\hat{\sigma}_{ps} = (\lambda\delta_{ps} - \gamma\xi_0\varepsilon_{ps}^0/I_1^0)\hat{\varepsilon}_{ii} + 2\mu_0^*\hat{\varepsilon}_{sp} + (-\gamma\xi_0\varepsilon_{mn}^0\delta_{ps}/I_1^0 + (\gamma + 3\delta\eta_0)\xi_0\varepsilon_{sp}^0\varepsilon_{mn}^0/I_2^0$$
$$- 3(\varepsilon_{sp}^0\varepsilon_{mq}^0\varepsilon_{qn}^0 + \varepsilon_{qp}^0\varepsilon_{sq}^0\varepsilon_{mn}^0)\,\delta\eta_0\xi_0/I_3^0)\hat{\varepsilon}_{mn} + 3(\delta\xi_0/I_1^0)(\hat{\varepsilon}_{sq}\varepsilon_{qp}^0 + \hat{\varepsilon}_{pq}\varepsilon_{qs}^0). \tag{4.1}$$

The equations of motion

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \rho\,\frac{\partial^2 u_i}{\partial t^2},$$

by virtue of steadiness of σ_{ij}^0, yield

$$\frac{\partial\hat{\sigma}_{ps}}{\partial x_s} = \rho\,\frac{\partial^2\hat{u}_p}{\partial t^2}. \tag{4.2}$$

The solution is sought in the form $\hat{u}_p = \tilde{u}_p \exp i(\omega t - k_\alpha x_\alpha)$. As the values of $\hat{\varepsilon}_{mn}$ are small, we obtain

$$\hat{\varepsilon}_{pq} = \frac{1}{2}\Big(\frac{\partial\hat{u}_p}{\partial x_q} + \frac{\partial\hat{u}_q}{\partial x_p}\Big). \tag{4.3}$$

From Eqs. (4.1)–(4.3), we obtain equations for the displacements $\tilde{u}_p$ in the form

$$x\tilde{u}_p = a_{p\alpha}\tilde{u}_\alpha,$$

where

$$x = \rho\omega^2/k^2 - \mu + 0.5\gamma\xi_0 + 0.5\delta\xi_0\eta_0 - 1.5(\delta\xi_0/I_1^0)\varepsilon_k^0 n_k = \rho\omega^2/k^2 - x_0,$$

$$a_{p\alpha} = A_0 n_\alpha n_p + D\varepsilon_{\alpha p}^0 + (D - B)(\varepsilon_\alpha^0 n_p + \varepsilon_p^0 n_\alpha) + C\varepsilon_p^0\varepsilon_\alpha^0 - b(E_\alpha\varepsilon_p^0 + E_p\varepsilon_\alpha^0),$$

$$A_0 = \lambda + \mu_0^* = \lambda + \mu - 0.5\gamma\xi_0 - 0.5\delta\eta_0\xi_0,$$

$$x_0 = \mu - 0.5\gamma\xi_0 - 0.5\delta\xi_0\eta_0 - 1.5(\delta\xi_0/I_1^0)\varepsilon_k^0 n_k, \tag{4.4}$$

$$D = 1.5\delta\xi_0/I_1^0 = 1.5\delta/\sqrt{I_2^0}, \qquad B = \gamma\xi_0/I_1^0 = \gamma/\sqrt{I_2^0},$$

$$C = (\gamma + 3\delta\eta_0)\xi_0/I_2^0, \qquad b = 3\delta\xi_0\eta_0/I_3^0 = 2D/I_2^0,$$

$$\varepsilon_\alpha^0 = \varepsilon_{s\alpha}^0 n_s, \qquad E_\alpha = \varepsilon_{\beta s}^0 \varepsilon_{s\alpha}^0 n_\beta, \qquad n_\alpha = k_\alpha/k, \qquad k^2 = k_s k_s, \qquad n_\alpha^2 = 1.$$

The characteristic equation

$$|a_{p\alpha} - x\delta_{p\alpha}| = 0$$

reduces to the cubic equation

$$x^3 - A_1 x^2 + A_2 x - A_3 = 0, \tag{4.5}$$

where

$$A_1 = a_{\alpha\alpha} = a_{11} + a_{22} + a_{33},$$

$$A_2 = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13}, \qquad A_3 = |a_{p\alpha}|.$$

Note, if $\delta = 0$, then $b = 0$, $D = 0$, and $A_3 = 0$.

By means of replacement of variables with allowance for Eq. (4.4), the cubic equation (4.5) is reduced to the form

$$z^3 + 3pz + 2q = 0, \qquad x = (z + \alpha_1/3)x_0,$$

$$p = (3\alpha_2 - \alpha_1^2)/9, \qquad q = (-2\alpha_1^3 + 9\alpha_1\alpha_2 - 27\alpha_3)/54,$$

where $\alpha_1 = A_1/x_0$, $\alpha_2 = A_2/x_0^2$, and $\alpha_3 = A_3/x_0^3$.

For small values of $|\delta/x_0|$, which is valid for $\delta \ll \mu$, the discriminant of the cubic equation is proportional to $|\delta/x_0|^3/108$; therefore, we can assume that $p^3 + q^2 \approx 0$. In this case, Eq. (4.5) is easily solved:

$$\begin{gathered} z_1 = 2(-q)^{1/3}, \qquad z_2 = z_3 = q^{1/3}, \\ x_i = (\rho\omega^2/k^2)_i - x_0 = x_0(z_i + \alpha_1/3), \qquad V_i = \sqrt{x_0(z_i + 1 + \alpha_1/3)/\rho} \quad (i = 1, 2, 3). \end{gathered} \tag{4.6}$$

If there is no preliminary deformation ($\varepsilon_{ij}^0 = 0$), we have

$$a_{p\alpha} = A_0 n_p n_\alpha, \qquad A_0 = \lambda + \mu, \qquad A_1 = A_0, \qquad A_2 = A_3 = 0,$$

$$x_1 = x_2 = 0, \qquad x_3 = A_0, \qquad x_0 = \mu,$$

$$V_p = \sqrt{(\lambda + 2\mu)/\rho}, \qquad V_s = \sqrt{\mu/\rho}.$$

We have $|\delta/x_0| = 0.044$ for St. 40 steel (normalized), $|\delta/x_0| = 0.172$ for SCh 12-28 cast iron, and $|\delta/x_0| = 0.014$ for AL-2 silumin. Thus, we can use $p^3 + q^2 \approx 0$ for the materials considered.

5. Now we consider uniform strain tension (compression) as the initial state. Taking into account Eq. (4.4), we obtain

$$\varepsilon_{ij}^0 = \pm\varepsilon\delta_{ij}, \quad \varepsilon \geqslant 0, \qquad a_{ps} = \pm(\sqrt{3}\,\delta/2)\delta_{ps} + (\lambda + \mu \mp 5\sqrt{3}\gamma/6 \pm \sqrt{3}\,\delta/2)n_p n_s,$$

$$x = \rho\omega^2/k^2 - x_0, \qquad x_0 = \mu \mp \sqrt{3}\,\gamma/2 \pm \sqrt{3}\,\delta/3,$$

$$\alpha_1 = \frac{\lambda + \mu}{x_0} \pm \sqrt{3}\Big(2 - \frac{5}{6}\,\frac{\gamma}{\delta}\Big)\frac{\delta}{x_0}, \qquad \alpha_2 = \Big[\pm\sqrt{3}\,\frac{\lambda + \mu}{x_0} + \frac{5}{2}\Big(\frac{3}{2} - \frac{\gamma}{\delta}\Big)\frac{\delta}{x_0}\Big]\frac{\delta}{x_0},$$

$$\alpha_3 = \Big[\frac{3}{4}\,\frac{\lambda + \mu}{x_0} \pm \frac{\delta}{x_0}\,\frac{\sqrt{3}}{4}\Big(3 - \frac{5}{2}\,\frac{\gamma}{\delta}\Big)\Big]\Big(\frac{\delta}{x_0}\Big)^2.$$

The upper sign here (plus or minus) refers to tension, and the lower sign refers to compression. As $|\delta/x_0|$ is small, Eq. (4.6) yields

$$V_1 = \sqrt{x_0(q^{1/3} + 1 + \alpha_1/3)/\rho}, \qquad V_2 = V_3 = \sqrt{x_0(-2q^{1/3} + 1 + \alpha_1/3)/\rho}.$$

Phase velocities for three materials with allowance for preliminary uniform strain tension or compression are listed in Table 2. The values of the phase velocities are independent of the direction of the wave vector.

TABLE 2

Longitudinal and Transverse Velocities
for Three Material under Uniform Strain Tension and Compression

Material	v^1_{tens}, m/sec	v^2_{tens}, m/sec	v^1_{compr}, m/sec	v^2_{compr}, m/sec
St. 40 steel	3186	5885	3218	5939
SCh 12-28 cast iron	2050	3916	2784	4635
AL-2 silumin	2837	6428	3380	7064

6. Now we consider strain shear as the initial state: $\varepsilon_{11} = \varepsilon$, $\varepsilon_{22} = -\varepsilon$, and the remaining values are $\varepsilon_{ij} = 0$. Then, we obtain

$$I_1^0 = 0, \qquad I_2^0 = 2\varepsilon^2, \qquad I_3^0 = 0, \qquad \xi_0 = 0, \qquad \eta_0 \neq 0,$$

$$B = \sqrt{2}\,\gamma/(2\varepsilon) = \tilde{B}/\varepsilon, \quad D = 3\sqrt{2}\,\delta/(4\varepsilon) = \tilde{D}/\varepsilon, \quad C = 0, \quad b = \tilde{D}/\varepsilon^3, \quad A_0 = \lambda + \mu,$$

$$a_{11} = \tilde{D} + (A_0 - 2\tilde{B})n_1^2, \qquad a_{22} = -\tilde{D} + (A_0 + 2\tilde{B})n_2^2, \qquad a_{33} = A_0 n_3^2,$$

$$a_{12} = A_0 n_1 n_2, \qquad a_{13} = (A_0 + \tilde{D} - \tilde{B})n_1 n_3, \qquad a_{23} = (A_0 - \tilde{D} + \tilde{B})n_2 n_3,$$

$$x = \rho\omega^2/k^2 - \mu + \tilde{D}(n_2^2 - n_1^2), \qquad x_0 = \mu - \tilde{D}(n_2^2 - n_1^2).$$

$$A_1 = A_0 + 2\tilde{B}(n_2^2 - n_1^2),$$

$$A_2 = -\tilde{D}^2 + \tilde{D}A_0(n_2^2 - n_1^2)(1 - 2n_3^2) + 2\tilde{D}\tilde{B}(n_2^2 + n_1^2) - 4\tilde{B}^2 n_2^2 n_1^2 - (\tilde{D} - \tilde{B})^2(n_2^2 + n_1^2)n_3^2,$$

$$A_3 = [A_0\tilde{D}^2(-1 + 2n_1^2 + 2n_2^2) - \tilde{D}(\tilde{D} - \tilde{B})^2(n_2^2 - n_1^2) - 4A_0\tilde{D}^2 n_2^2 n_1^2]n_3^2.$$

In this case, the velocity distribution becomes anisotropic owing to the initial strain (the velocities depend on the wave vector).

For an arbitrary vector with $n_3 = 0$ and $n_1^2 + n_2^2 = 1$, we obtain $a_{33} = a_{13} = a_{23} = 0$, $A_3 = 0$, $x_0 = \mu - \tilde{D}(n_2^2 - n_1^2)$. In this case, the characteristic equation has the following roots: $x_1 = 0$ and $x_{2,3} = (A_1 \pm \sqrt{A_1^2 - 4A_2})/2$. Thereby,

$$A_1^2 - 4A_2 = (a_{11} - a_{22})^2 + 4a_{12}^2 = [A_0 - 2(\tilde{D} - \tilde{B})]^2 + 8A_0(\tilde{D} - \tilde{B})n_1^2.$$

For $n_1 = 1$ and $n_2 = n_3 = 0$, we obtain the following relations for the wave vectors directed along the coordinate axes:

$$x_0 = \mu - \tilde{D}, \qquad A_1^2 - 4A_2 = (A_0 - 2\tilde{B} + 2\tilde{D})^2.$$

It follows from here that

$$V_1 = \sqrt{(\mu + \tilde{D})/\rho} = \sqrt{(\mu + 3\sqrt{2}\,\delta/4)/\rho}, \qquad V_2 = \sqrt{\mu/\rho},$$

$$V_3 = \sqrt{(\lambda + 2\mu + 2\tilde{D} - 2\tilde{B})/\rho} = \sqrt{(\lambda + 2\mu + 3\sqrt{2}\,\delta/2 - \sqrt{2}\,\gamma)/\rho}.$$

For $n_1 = 0$, $n_2 = 1$, and $n_3 = 0$, we have

$$x_0 = \mu + \tilde{D}, \qquad A_1^2 - 4A_2 = (A_0 + 2\tilde{B} - 2\tilde{D})^2,$$

$$V_1 = \sqrt{(\mu - \tilde{D})/\rho} = \sqrt{(\mu - 3\sqrt{2}\,\delta/4)/\rho}, \qquad V_2 = \sqrt{\mu/\rho},$$

$$V_3 = \sqrt{(\lambda + 2\mu - 2\tilde{D} + 2\tilde{B})/\rho} = \sqrt{(\lambda + 2\mu - 3\sqrt{2}\,\delta/2 + \sqrt{2}\,\gamma)/\rho}.$$

TABLE 3

Longitudinal and Transverse Velocities for Three Materials under Shear

Material	$\boldsymbol{n} = (1, 0, 0)$			$\boldsymbol{n} = (0, 1, 0)$		
	v^1, m/sec	v^2, m/sec	v^3, m/sec	v^1, m/sec	v^2, m/sec	v^3, m/sec
St. 40 steel	5889.0	3182.70	3208.6	6001.30	3335.76	3310.90
SCh 12-28 cast iron	4000.7	2263.10	2534.3	4571.23	2722.10	2471.57
AL-2 silumin	6712.7	3128.75	3335.7	6887.63	3157.70	2957.60

TABLE 4

Calculated Longitudinal and Transverse Velocities under Pure Uniaxial Tension and Pure Uniaxial Compression

Material	$\sqrt{\mu/\rho}$	$\sqrt{\mu_{\text{tens}}/\rho}$	$\sqrt{\mu_{\text{compr}}/\rho}$	$\sqrt{(\lambda+2\mu)/\rho}$	$\sqrt{(\lambda_{\text{tens}}+2\mu_{\text{tens}})/\rho}$	$\sqrt{(\lambda_{\text{compr}}+2\mu_{\text{compr}})/\rho}$
St. 40 steel	3260	3228.5	3289.8	5945.7	5939.7	5951.5
SCh 12-28 cast iron	2503	2336.8	2644.7	4295.5	3900.3	4711.6
AL-2 silumin	3152	3101.2	3189.5	6822.3	6298.5	7510.1

For $n_1 = 0$, $n_2 = 0$, and $n_3 = 1$, we obtain

$$x_0 = \mu, \qquad A_1 = A_0, \qquad A_2 = -\tilde{D}^2, \qquad A_3 = -A_0\tilde{D}^2,$$

$$V_1 = \sqrt{(\mu - \tilde{D})/\rho}, \qquad V_2 = \sqrt{(\mu + \tilde{D})/\rho}, \qquad V_3 = \sqrt{(\lambda + 2\mu)/\rho}.$$

The values of the longitudinal and transverse velocities calculated for three materials under shear with the wave vector in the plane $x_3 = 0$ are listed in Table 3. The results are given for two wave vectors: $\boldsymbol{n} = (1, 0, 0)$ and $\boldsymbol{n} = (0, 1, 0)$.

For comparison, Table 4 gives the calculated velocities for the case with $\delta = 0$ and $\gamma = 0$ and for constants obtained in experiments on pure uniaxial tension and pure uniaxial compression.

Conclusions. The tensor-nonlinear model including four material constants, which is described in the paper, allows determining phase velocities for various types of preliminary loading of the material with allowance the multimodulus character of this material. The model offers a possibility of removing restrictions on relations between the constants, which are inevitable in simpler models. The velocity of sound, even for homogeneous and isotropic materials, is demonstrated to depend not only on the stress-strain state and on the moduli difference characterized by two terms with the coefficients γ and δ, but also on the direction of sonic wave propagation. The magnitude of the difference in velocities depends substantially on the difference in Poisson's ratios under tension and compression.

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